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On the quantum statistics of the superposition of coherent and chaotic fields

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Abstract. The generating function, the corresponding integrated intensity distribution, and the photon-counting distribution and its factorial moments are derived for the superposition of coherent and chaotic M-mode fields on the basis of a formalism of arbitrary ordering of field operators in quantum optics. An alternative description is also given based on a recent analysis of the superposition of coherent and chaotic fields in terms of the correlation functions, which makes it possible to derive a number of results valid for arbitrary mean occupation numbers per mode from which the model considered follows as a special case. Many earlier results obtained for coherent and chaotic fields and their superposition are included as special cases and some of them complete a table recently published by Jakeman and Pike summarizing results obtained in this field. In particular the photon-counting distribution and its factorial moments are given for the superposition of coherent and chaotic fields with different mean frequencies, which is important in connection with the statistics of heterodyne detection of thermal light. The present results generally describe a superposition experiment for an N-mode coherent field (e.g. generated by an N-mode laser operating far above threshold) and an M-mode chaotic field ($M \geqslant N$) as well as a field generated by an M-mode laser operating above threshold.

1. Introduction

The statistics of the superposition of coherent and chaotic fields has been studied theoretically by a number of authors because of its importance for the description of laser light above the threshold of oscillations. The first papers dealing with this subject were published by Lachs (1965) and by Glauber (1966) in which the photon-counting distribution and its factorial moments for the superposition of one-mode coherent and chaotic fields with the same frequencies were given. Another approach to this problem, based on the calculation of the correlation function, was proposed by Morawitz (1965) for one-mode fields. Peřina (1967, 1968 a, b) generalized these results to the superposition of M-mode coherent and chaotic fields with overlapping frequencies; in this analysis expressions for anti-normally ordered field operators have also been included. A more general description of the superposition of such fields, based on the description of fields by means of correlation functions generalizing the results of Morawitz (1965, 1966), the results of which are also valid for M-mode fields, was proposed by Peřina and Mišta (1968). Such an approach represents a general description of the superposition of coherent and chaotic fields and enables us to derive all results for the model with different mean frequencies considered below as a special case. Some recent investigations of the problem of ordering of field operators in quantum optics and correspondence between functions of c numbers and functions of operators (q numbers), investigated by Agarwal and Wolf (1968), by Wolf and Agarwal (1969) (for another approach see also Lax 1968) and further by Cahill and Glauber (1969 a, b), make it possible to derive very general formulae describing the statistical properties of the superposition of coherent and chaotic fields with respect to arbitrary ordering of field operators, which was done by Peřina and Horák (1969 a, b, to be referred to as PH). In particular, formulae concerning anti-normal ordering of field operators could have great importance in connection with detection of optical fields by means of the so-called quantum counters operating by stimulated emission rather than by absorption (Mandel 1966, Glauber, to be published).

In a recent paper by Jakeman and Pike (1969) it was pointed out that additional spectral information can be obtained by the use of heterodyne detection, where the chaotic field is superimposed, before detection, on a known coherent component. In this way the centre frequency of the chaotic field can be determined. However, the general heterodyne problem for chaotic (thermal) light includes the case when both the centre frequency of the chaotic field and the frequency of the coherent field are arbitrary and differ from one another. That is why these authors studied the superposition of one-mode chaotic and coherent fields with different frequencies in greater detail. In particular they derived the corresponding expressions for the generating function, the intensity distribution and the second factorial moment of the photon-counting distribution, but they did not give the photon-counting distribution, which is an important measurable quantity, and its arbitrary-order factorial moments. All results obtained to date for the superposition of one-mode coherent and chaotic fields (and for the coherent and chaotic fields alone as special cases) by these authors as well as by a number of other authors have been arranged in table 1 of the paper by Jakeman and Pike (1969, Pike, to be published).

The main purpose of this paper is to complete Jakeman and Pike's table for the superposition of one-mode coherent and narrow-band chaotic fields with different frequencies, i.e. to derive the photon-counting distribution and its factorial moments of arbitrary order for this case. However, using our earlier results concerning M-mode fields and the general formalism of arbitrary ordering of field operators mentioned above we shall be able not only to complete the table but also to generalize all results in the table concerning narrow-band chaotic fields, coherent fields and their superposition (the third, fifth, seventh and ninth lines of the table) to M-mode fields and to arbitrary ordering of field operators, i.e. to derive the generating function, the integrated intensity distribution and its moments, and the photon-counting distribution and its factorial moments for this case. Furthermore, we show that all these results follow as special cases from a slight extension of an analysis by Peřina and Mišta (1968), carried out in terms of the correlation functions. Of course, such an analysis can also serve for derivation of equations describing fields with an arbitrary spectrum (cf. the first line of the table). Present results generally describe a superposition experiment for an N-mode coherent field (e.g. generated by an N-mode laser operating far above threshold) and an M-mode chaotic field ($M \geqslant N$) as well as a field generated by an M-mode laser operating above threshold.

In § 2, the s-ordered form of the generating function for M-mode fields is derived using the formalism proposed by PH and the corresponding integrated intensity distribution is given. In § 3 the photon-counting distribution and its factorial moments as well as the s-ordered moments of the integrated intensity distribution related to arbitrary ordering are obtained. An alternative description is also given based on a recent analysis of the superposition of coherent and chaotic fields in terms of the correlation functions enabling us to derive a number of results valid for arbitrary mean occupation numbers per mode from which the model considered follows as a special case. In § 4 the correspondence with earlier results is discussed. In § 5 some perspectives are outlined.

2. s-ordered generating function and integrated intensity distribution

The one-mode generating function for the considered model of the superposition of coherent and chaotic fields was calculated by Jakeman and Pike (1969) by the use of the Fredholm determinant of a homogeneous integral equation†

$$\langle \exp(\mathrm{i}x\hat{n}) \rangle_N = (1-\mathrm{i}x\langle n_\mathrm{T}\rangle)^{-1} \exp\left\{\frac{\mathrm{i}x\omega^2\langle n_\mathrm{C}\rangle}{1-\mathrm{i}x\langle n_\mathrm{T}\rangle} + \mathrm{i}x\langle n_\mathrm{C}\rangle(1-\omega^2)\right\}$$

$$= (1-\mathrm{i}x\langle n_\mathrm{T}\rangle)^{-1} \exp\left\{\frac{(\mathrm{i}x)^2\omega^2\langle n_\mathrm{C}\rangle\langle n_\mathrm{T}\rangle}{1-\mathrm{i}x\langle n_\mathrm{T}\rangle} + \mathrm{i}x\langle n_\mathrm{C}\rangle\right\} \quad (2.1)$$

where $\hat{n}$ is the photon number operator, $\langle n_\mathrm{T}\rangle$ and $\langle n_\mathrm{C}\rangle$ are the mean occupation numbers of

† For our purpose Jakeman and Pike's generating function $Q(s)$ is slightly modified: $\langle \exp(\mathrm{i}x\hat{n})\rangle_N = Q(-\mathrm{i}x)$. We also put $\alpha = 1$ for the photoefficiency α so that $\langle \hat{n} \rangle = \alpha\langle W\rangle_N = \langle W\rangle_N$, where W is the integrated intensity.

photons in the chaotic and the coherent fields, respectively, and $\omega = 2\sin(\frac{1}{2}\Omega)/\Omega$, where Ω represents the difference between the mean frequency of the chaotic field and the frequency of the coherent field multiplied by the time interval T of the observation. The subscript N expresses the fact that this generating function is normally ordered. We see that the one-mode generating function (2.1) can be interpreted as a product of the generating function describing the superposition of chaotic and coherent fields with mean occupation numbers $\langle n_{\mathrm{T}}\rangle$ and $\langle n_{\mathrm{C}}\rangle$, ω^2 and the generating function describing a purely coherent field with a mean occupation number $\langle n_{\mathrm{C}}\rangle(1-\omega^2)$.

We can consider a more general case, i.e. the superposition of a coherent field which is in a coherent state $\Pi_{\lambda=1}^{M}|\beta_\lambda\rangle$ (we assume that $\beta_\lambda = 0$ for $\lambda = M+1, \ldots$) with the diagonal representation weighting function

$$\phi_{\mathrm{C}}(\{\alpha_\lambda\}) = \prod_{\lambda=1}^{M} \delta(\alpha_\lambda - \beta_\lambda) \tag{2.2}$$

and a chaotic M-mode field described by

$$\phi_{\mathrm{T}}(\{\alpha_\lambda\}) = \prod_{\lambda=1}^{M} (\pi\langle n_{\mathrm{T}\lambda}\rangle)^{-1} \exp\left(-\frac{|\alpha_\lambda|^2}{\langle n_{\mathrm{T}\lambda}\rangle}\right). \tag{2.3}$$

The superposition of these two fields is described by a general formula (9.15) of Glauber (1965) expressing the quantum superposition principle

$$\begin{aligned}\phi(\{\alpha_\lambda\}) &= \int \prod_{\lambda}^{M} \delta(\alpha_\lambda - \alpha_\lambda' - \alpha_\lambda'')\phi_{\mathrm{C}}(\{\alpha_\lambda'\})\phi_{\mathrm{T}}(\{\alpha_\lambda''\}) \prod_{\lambda}^{M} \mathrm{d}^2\alpha_\lambda'\, \mathrm{d}^2\alpha_\lambda'' \\ &= \prod_{\lambda}^{M} (\pi\langle n_{\mathrm{T}\lambda}\rangle)^{-1} \exp\left(-\frac{|\alpha_\lambda - \beta_\lambda|^2}{\langle n_{\mathrm{T}\lambda}\rangle}\right).\end{aligned} \tag{2.4}$$

Thus the generating function becomes

$$\begin{aligned}\langle \exp(\mathrm{i}x\hat{n})\rangle_N &= \int \prod_{\lambda}^{M} (\pi\langle n_{\mathrm{T}\lambda}\rangle)^{-1} \exp\left(-\frac{|\alpha_\lambda - \beta_\lambda|^2}{\langle n_{\mathrm{T}\lambda}\rangle} + \mathrm{i}xW\right) \prod_{\lambda}^{M} \mathrm{d}^2\alpha_\lambda \\ &= \prod_{\lambda}^{M} (1 - \mathrm{i}x\langle n_{\mathrm{T}\lambda}\rangle)^{-1} \exp\left(\frac{\mathrm{i}x\langle n_{\mathrm{C}\lambda}\rangle}{1 - \mathrm{i}x\langle n_{\mathrm{T}\lambda}\rangle}\right)\end{aligned} \tag{2.5}$$

where

$$W = \sum_{\lambda}^{M} |\alpha_\lambda|^2 = \langle\{\alpha_\lambda\}| \int_{L^3} \hat{A}^+(x)\hat{A}(x)\, \mathrm{d}^3\boldsymbol{x} |\{\alpha_\lambda\}\rangle.$$

Here $\hat{A}$ is the detection operator.

Now considering the $2M$-mode generating function (2.5) with mean occupation numbers $\langle n_{\mathrm{T}\lambda}\rangle = \langle n_{\mathrm{T}}\rangle/M$ per mode, $\lambda = 1, 2, \ldots, M$ and $\langle n_{\mathrm{T}\lambda}\rangle = 0$, $\lambda = M+1, \ldots, 2M$, where $\langle n_{\mathrm{T}}\rangle$ is the mean occupation number of photons in the whole chaotic field (results for fields with unequal mean occupation numbers per mode will be given later), we can easily obtain from (2.5) the normal generating function for the superposition of M-mode coherent and chaotic fields with frequency shifts, described by ω_λ, between modes λ and $\lambda+M$ by writing $\omega_\lambda^2\langle n_{\mathrm{C}\lambda}\rangle$ instead of $\langle n_{\mathrm{C}\lambda}\rangle$, $\lambda = 1, \ldots, M$ and $\langle n_{\mathrm{C}\lambda}\rangle(1-\omega_\lambda^2)$ instead of $\langle n_{\mathrm{C}\lambda+M}\rangle$:

$$\begin{aligned}\langle \exp(\mathrm{i}x\hat{n})\rangle_N &= \prod_{\lambda}^{M}\left(1 - \mathrm{i}x\frac{\langle n_{\mathrm{T}}\rangle}{M}\right)^{-1} \exp\left\{\frac{\mathrm{i}x\omega_\lambda^2\langle n_{\mathrm{C}\lambda}\rangle}{1 - \mathrm{i}x\langle n_{\mathrm{T}}\rangle/M} + \mathrm{i}x\langle n_{\mathrm{C}\lambda}\rangle(1-\omega_\lambda^2)\right\} \\ &= \left(1 - \mathrm{i}x\frac{\langle n_{\mathrm{T}}\rangle}{M}\right)^{-M} \exp\left\{\frac{\mathrm{i}x\omega^2\langle n_{\mathrm{C}}\rangle}{1 - \mathrm{i}x\langle n_{\mathrm{T}}\rangle/M} + \mathrm{i}x\langle n_{\mathrm{C}}\rangle(1-\omega^2)\right\} \\ &= \left(1 - \mathrm{i}x\frac{\langle n_{\mathrm{T}}\rangle}{M}\right)^{-M} \exp\left(\frac{(\mathrm{i}x)^2\omega^2\langle n_{\mathrm{C}}\rangle\langle n_{\mathrm{T}}\rangle}{M(1 - \mathrm{i}x\langle n_{\mathrm{T}}\rangle/M)} + \mathrm{i}x\langle n_{\mathrm{C}}\rangle\right)\end{aligned} \tag{2.6}$$

where $\omega^2\langle n_C\rangle = \Sigma_\lambda^M \omega_\lambda^2 \langle n_{C\lambda}\rangle$ $(\omega^2 = \Sigma_\lambda^M \omega_\lambda^2\langle n_{C\lambda}\rangle/\Sigma_\lambda^M \langle n_{C\lambda}\rangle)$ and $\langle n_C\rangle = \Sigma_\lambda^M \langle n_{C\lambda}\rangle$. For $M = 1$ we again obtain the one-mode generating function (2.1).

In general the generating function (2.5) as well as the special case (2.6) describes the superposition of an M-mode chaotic field with an N-mode coherent field ($M \geqslant N$). The coherent N-mode field can be generated by a laser operating on N modes far above the threshold, where it is approximately in the coherent state $|\{\beta_\lambda\}\rangle$, or by N one-mode lasers operating in this region, fields of which are superposed so that the resulting field is again in the coherent state $|\{\beta_\lambda\}\rangle$. But such a generating function also describes the statistical properties of light of an M-mode laser operating above the threshold where the model of the superposition of M-mode chaotic and M-mode coherent fields is appropriate. Also scattering experiments using an M-mode laser in this region will be described by the present formulae. In principle the generating function (2.6) is suitable for the description of heterodyne detection of a chaotic M-mode field, where the chaotic field is superimposed on a coherent N-mode field ($M \geqslant N$) with frequency shifts of the corresponding modes characterized by ω_λ. However, we see that the generating function (2.6) is of the same form independent of the number $N \leqslant M$ of the coherent modes. Hence, the practically more important case of detection of an M-mode chaotic field superimposed on a one-mode coherent field will be described by the generating function (2.6) and by all formulae following from it given below, putting $\langle n_{C\lambda}\rangle = 0$ for all modes except one.

In order to obtain the s-ordered M-mode generating function we use an equation of PH enabling us to express the s-ordered generating function by means of the normal generating function in the form

$$\langle \exp(\mathrm{i}x\hat{n})\rangle_s = \left(1-\frac{1-s}{2}\mathrm{i}x\right)^{-M}\left\langle \exp\left\{\frac{\mathrm{i}x\hat{n}}{1-\frac{1}{2}(1-s)\mathrm{i}x}\right\}\right\rangle_N, \quad -1 \leqslant s \leqslant 1. \tag{2.7}$$

Substituting (2.6) into (2.7) we arrive at

$$\begin{aligned}\langle \exp(\mathrm{i}x\hat{n})\rangle_s &= \left\{1-\mathrm{i}x\left(\frac{\langle n_T\rangle}{M}+\frac{1-s}{2}\right)\right\}^{-M}\exp\left[\frac{\mathrm{i}x\omega^2\langle n_C\rangle}{1-\mathrm{i}x\{\langle n_T\rangle/M+\frac{1}{2}(1-s)\}}\right.\\ &\quad\left.+\frac{\mathrm{i}x\langle n_C\rangle(1-\omega^2)}{1-\frac{1}{2}(1-s)\mathrm{i}x}\right]\\ &= \left\{1-\mathrm{i}x\left(\frac{\langle n_T\rangle}{M}+\frac{1-s}{2}\right)\right\}^{-M}\\ &\quad\times\exp\left(\frac{(\mathrm{i}x)^2\omega^2\langle n_C\rangle\langle n_T\rangle}{M\{1-\frac{1}{2}(1-s)\mathrm{i}x\}[1-\mathrm{i}x\{\langle n_T\rangle/M+\frac{1}{2}(1-s)\}]}\right.\\ &\quad\left.+\frac{\mathrm{i}x\langle n_C\rangle}{1-\frac{1}{2}(1-s)\mathrm{i}x}\right).\end{aligned} \tag{2.8}$$

If we compare (2.8) with (2.6) we see that while the generating function (2.6) includes one pole $(-\mathrm{i}M/\langle n_T\rangle)$ only the generating function (2.8) includes two poles $(-\mathrm{i}/\{\langle n_T\rangle/M+\frac{1}{2}(1-s)\}$ and $-2\mathrm{i}/(1-s))$. From this we can conclude that the integrated intensity distribution related to the s ordering of field operators for $s \neq 1$ and $\omega \neq 1$ will be qualitatively different from the integrated intensity distribution related to the normal ordering of field operators. On the other hand if $\omega = 1$ (the mean frequencies of the coherent and chaotic fields are the same) all these distributions are of the same form, as one can see from the corresponding generating function (PH)

$$\langle \exp(\mathrm{i}x\hat{n})\rangle_s = \left\{1-\mathrm{i}x\left(\frac{\langle n_T\rangle}{M}+\frac{1-s}{2}\right)\right\}^{-M}\exp\left[\frac{\mathrm{i}x\langle n_C\rangle}{1-\mathrm{i}x\{\langle n_T\rangle/M+\frac{1}{2}(1-s)\}}\right] \tag{2.9}$$

having one pole $(-\mathrm{i}/\{\langle n_T\rangle/M+\frac{1}{2}(1-s)\})$ for all s $(-1 \leqslant s \leqslant 1)$. The generating function (2.8) can be decomposed in a double series of the powers of $\mathrm{i}x$ with coefficients proportional to the Laguerre polynomials. This can serve, in principle, for determination, for example, of the s-ordered moments

$$\langle \hat{n}^k \rangle_s = \frac{\mathrm{d}^k}{\mathrm{d}(\mathrm{i}x)^k} \langle \exp(\mathrm{i}x\hat{n}) \rangle_s |_{\mathrm{i}x=0}$$

but there are some difficulties in doing this. Consequently we shall use another method of determining this and other quantities later.

In order to determine the distribution $P(W, s)$ of the integrated intensity W related to the s-ordered operators we use another result of PH expressing $P(W, s)$ by means of $P(W, 1) \equiv P_N(W)$:

$$P(W, s) = \frac{2}{1-s} \int_0^\infty \left(\frac{W}{W'}\right)^{(M-1)/2} \exp\left\{-\frac{2(W+W')}{1-s}\right\} \mathrm{I}_{M-1}\left(4\frac{(WW')^{1/2}}{1-s}\right) P_N(W')\,\mathrm{d}W' \tag{2.10}$$

where $\mathrm{I}_{M-1}(x)$ is the modified Bessel function. Distribution $P_N(W)$ can be calculated from the generating function (2.6) with the aid of a Fourier transform:

$$P_N(W) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle \exp(\mathrm{i}x\hat{n}) \rangle_N \exp(-\mathrm{i}xW)\,\mathrm{d}x$$

$$= \begin{cases} \dfrac{M}{\langle n_\mathrm{T} \rangle} \left(\dfrac{W - \langle n_\mathrm{C} \rangle (1-\omega^2)}{\langle n_\mathrm{C} \rangle \omega^2}\right)^{(M-1)/2} \exp\left\{-\dfrac{W + \langle n_\mathrm{C} \rangle (2\omega^2 - 1)}{\langle n_\mathrm{T} \rangle} M\right\} & \\ \times I_{M-1}\left(2\omega M \dfrac{[\langle n_\mathrm{C} \rangle \{W - \langle n_\mathrm{C} \rangle (1-\omega^2)\}]^{1/2}}{\langle n_\mathrm{T} \rangle}\right) & W > \langle n_\mathrm{C} \rangle (1-\omega^2) \\ 0 \qquad W < \langle n_\mathrm{C} \rangle (1-\omega^2) & \end{cases} \tag{2.11}$$

where the residuum theorem was used. Substituting this result into (2.10), after some mathematics we obtain, writing the I_{M-1} function in the form of the series,

$$\begin{aligned} P(W, s) = {} & \left\{\frac{W}{\langle n_\mathrm{C} \rangle (1-\omega^2)}\right\}^{(M-1)/2} \left(\frac{\langle n_\mathrm{T} \rangle}{M} + \frac{1-s}{2}\right)^{-M} \left(\frac{1-s}{2}\right)^{M-1} \\ & \times \exp\left[-\frac{2W}{1-s} - \frac{\langle n_\mathrm{C} \rangle}{\langle n_\mathrm{T} \rangle / M + \frac{1}{2}(1-s)} - \frac{2 \langle n_\mathrm{C} \rangle \langle n_\mathrm{T} \rangle (1-\omega^2)}{M(1-s)\{\langle n_\mathrm{T} \rangle / M + \frac{1}{2}(1-s)\}}\right] \\ & \times \sum_{l=0}^{\infty} \frac{1}{(l+M-1)!} \left[\left\{\frac{W}{\langle n_\mathrm{C} \rangle (1-\omega^2)}\right\}^{1/2} \frac{\langle n_\mathrm{T} \rangle}{M} \left(\frac{\langle n_\mathrm{T} \rangle}{M} + \frac{1-s}{2}\right)^{-1}\right]^l \\ & \times \mathrm{I}_{l+M-1}\left[4\frac{\{W \langle n_\mathrm{C} \rangle (1-\omega^2)\}^{1/2}}{1-s}\right] \mathrm{L}_l^{M-1}\left[-\frac{(1-s)\omega^2 \langle n_\mathrm{C} \rangle M}{2 \langle n_\mathrm{T} \rangle \{\langle n_\mathrm{T} \rangle / M + \frac{1}{2}(1-s)\}}\right] \end{aligned} \tag{2.12}$$

where

$$\mathrm{L}_l^{M-1}(x) = \{(l+M-1)!\}^2 \sum_{j=0}^{l} \frac{(-x)^j}{j!(l-j)!(j+M-1)!}$$

is the Laguerre polynomial.

3. The photon-counting distribution, its factorial moments and the s-ordered moments of $P(W, s)$

The simplest way of obtaining the photon-counting distribution is to use the photodetection equation (Mandel 1958)

$$p(n) = \int_0^\infty \frac{W^n}{n!} \exp(-W) P_N(W)\,\mathrm{d}W. \tag{3.1}$$

Substituting (2.11) into (3.1), where the substitution $W = W' + \langle n_\mathrm{C} \rangle (1-\omega^2)$ is made and the binomial theorem used, we arrive at

$$p(n) = \exp\{-\langle n_C\rangle(1-\omega^2)\} \sum_{j=0}^{n} \binom{n}{j} \{\langle n_C\rangle(1-\omega^2)\}^{n-j} p'(j) \tag{3.2}$$

where $p'(j)$ is the photon-counting distribution

$$p'(j) = \frac{1}{(j+M-1)!}\left(1+\frac{M}{\langle n_T\rangle}\right)^{-j}\left(1+\frac{\langle n_T\rangle}{M}\right)^{-M} \times \exp\left(-\frac{\langle n_C\rangle\omega^2 M}{M+\langle n_T\rangle}\right) L_j^{M-1}\left\{-\frac{\langle n_C\rangle\omega^2 M^2}{\langle n_T\rangle(\langle n_T\rangle+M)}\right\} \tag{3.3}$$

corresponding to the distribution (Peřina 1967)

$$P_N'(W') = \frac{M}{\langle n_T\rangle}\left(\frac{W'}{\langle n_C\rangle\omega^2}\right)^{(M-1)/2} \exp\left(-\frac{W'+\langle n_C\rangle\omega^2}{\langle n_T\rangle}M\right) \times I_{M-1}\left(2\frac{\omega(W'\langle n_C\rangle)^{1/2}}{\langle n_T\rangle}M\right). \tag{3.4}$$

Hence the resulting photon-counting distribution is

$$p(n) = \left(1+\frac{\langle n_T\rangle}{M}\right)^{-M} \exp\left[-\frac{\langle n_C\rangle\{M+\langle n_T\rangle(1-\omega^2)\}}{M+\langle n_T\rangle}\right] \times \sum_{j=0}^{n} \frac{1}{(n-j)!(j+M-1)!}\{\langle n_C\rangle(1-\omega^2)\}^{n-j}\left(1+\frac{M}{\langle n_T\rangle}\right)^{-j} \times L_j^{M-1}\left\{-\frac{\langle n_C\rangle\omega^2 M^2}{\langle n_T\rangle(\langle n_T\rangle+M)}\right\}. \tag{3.5}$$

One can obtain the factorial moments of $p(n)$ (the moments of $P_N(W)$) in the same way if the moments of (3.4) (factorial moments of (3.3)) (Peřina 1967, 1968 a, b)

$$\langle W'^k\rangle_N = \left\langle\frac{n!}{(n-k)!}\right\rangle = \frac{k!}{(k+M-1)!}\left(\frac{\langle n_T\rangle}{M}\right)^k L_k^{M-1}\left(-\frac{\langle n_C\rangle\omega^2 M}{\langle n_T\rangle}\right) \tag{3.6}$$

are used. One obtains

$$\langle W^k\rangle_N = \left\langle\frac{n!}{(n-k)!}\right\rangle = \sum_{j=0}^{k} \frac{k!}{(k-j)!(j+M-1)!}\{\langle n_C\rangle(1-\omega^2)\}^{k-j}\left(\frac{\langle n_T\rangle}{M}\right)^j L_j^{M-1}\left(-\frac{\langle n_C\rangle\omega^2 M}{\langle n_T\rangle}\right). \tag{3.7}$$

Of course, equations (3.5) and (3.7) can be obtained from the generating function (2.6) taking into account that

$$(1-\mathrm{i}xB)^{-M} \exp\left[\frac{\mathrm{i}xA}{1-\mathrm{i}xB}\right] = \sum_{n=0}^{\infty} \frac{(\mathrm{i}xB)^n}{(n+M-1)!} L_n^{M-1}\left(-\frac{A}{B}\right) \tag{3.8a}$$

$$= (1+B)^{-M} \exp\left(-\frac{A}{1+B}\right) \times \sum_{n=0}^{\infty} \frac{1}{(n+M-1)!}\left(1+\frac{1}{B}\right)^{-n} \times (1+\mathrm{i}x)^n L_n^{M-1}\left(-\frac{A}{B(B+1)}\right). \tag{3.8b}$$

Using these identities for $\exp\{ix\omega^2\langle n_C\rangle/(1-ix\langle n_T\rangle/M)\}$, decomposing $\exp\{ix\langle n_C\rangle(1-\omega^2)\}$ in the Taylor series at $ix = 0$ or at $ix = -1$ and using

$$\langle W^k\rangle_N = \frac{d^k}{d(ix)^k}\langle \exp(ix\hat{n})\rangle_N|_{ix=0}$$

or

$$p(n) = (n!)^{-1}\frac{d^n}{d(ix)^n}\langle \exp\text{-}(ix\hat{n})\rangle_N|_{ix=-1}$$

we again arrive at (3.5) or (3.7).

We wish to point out here that expression (3.7) can easily be obtained from the expression (Peřina and Mišta 1968)

$$\langle W^k\rangle_N = \frac{d^k}{d(ix)^k}\prod_\lambda (1-ix\langle n_{T\lambda}\rangle)^{-1}\exp\left(\frac{ix\langle n_{C\lambda}\rangle}{1-ix\langle n_{T\lambda}\rangle}\right)\Bigg|_{ix=0}$$

$$= k!\sum_{\Sigma_\lambda m_\lambda = k}\prod_\lambda \frac{1}{m_\lambda!}\langle n_{T\lambda}\rangle^{m_\lambda} L^0_{m_\lambda}\left(-\frac{\langle n_{C\lambda}\rangle}{\langle n_{T_\lambda}\rangle}\right) \tag{3.9}$$

which is a result of a general analysis of the superposition of coherent and chaotic fields in terms of the correlation functions. As one can see from (2.6) the generating function is a product of the generating function of the M-mode superposition with mean occupation numbers $\langle n_T\rangle/M$ per mode in the chaotic field and $\langle n_{C\lambda}\rangle\omega_\lambda^2$ in the coherent field and the generating function of the purely coherent M-mode field with mean occupation numbers $\langle n_{C\lambda}\rangle(1-\omega_\lambda^2)$ per mode (the full occupation number in the coherent field is $\langle n_C\rangle\omega^2+\langle n_C\rangle(1-\omega^2) = \langle n_C\rangle$; hence the parameter ω distributes the mean occupation number $\langle n_C\rangle$ between the coherent part of the superposition and the purely coherent field. For $\omega = 1$ ($\Omega = 0$) $\langle n_C\rangle$ belongs fully to the superposition; for $\omega = 0$ ($\Omega = \infty$ or the zeros of the function $2\sin\frac{1}{2}\Omega/\Omega$) $\langle n_C\rangle$ belongs fully to the purely coherent field and we have a product of the generating functions of the purely chaotic and purely coherent fields in this case. Therefore we obtain from (3.9), putting $\langle n_{T\lambda}\rangle = 0$ for $\lambda = M+1, \ldots, 2M$ and $\langle n_{T\lambda}\rangle = \langle n_T\rangle/M$ for $\lambda = 1, \ldots, M$ and using the asymptotic expression for L_m,

$$\langle W^k\rangle_N = k!\sum_{\Sigma^{2M}_\lambda m_\lambda = k}\left(\frac{\langle n_T\rangle}{M}\right)^{\sum^M_\lambda m_\lambda}\prod^M_\lambda \frac{1}{m_\lambda!}L^0_{m_\lambda}\left(-\frac{\langle n_{C\lambda}\rangle M}{\langle n_T\rangle}\right)\prod^{2M}_{\lambda=M+1}\frac{\langle n_{C\lambda}\rangle^{m_\lambda}}{m_\lambda!}. \tag{3.10}$$

Writing

$$\sum_{\Sigma^{2M}_\lambda m_\lambda = k} = \sum^k_{j=0}\sum_{\Sigma^M_\lambda m_\lambda = j}\sum_{\Sigma^{2M}_{\lambda=M+1} m_\lambda = k-j}$$

and using the following identity for the Laguerre polynomials

$$\sum_{\Sigma^M_\lambda m_\lambda = j}\prod^M_\lambda \frac{1}{(m_\lambda+\alpha_\lambda)!}L^{\alpha_\lambda}_{m_\lambda}(X_\lambda) = \frac{1}{(j+\Sigma^M_\lambda\alpha_\lambda+M-1)!}L_j^{\Sigma^M_\lambda\alpha_\lambda+M-1}\left(\sum^M_\lambda X_\lambda\right) \tag{3.11}$$

and the polynomial theorem we again arrive at (3.7), where

$$\sum^M_\lambda \langle n_{C\lambda}\rangle \to \sum^M_\lambda \langle n_{C\lambda}\rangle\omega_\lambda^2 = \langle n_C\rangle\omega^2$$

and

$$\sum_{\lambda=M+1}^{2M} \langle n_{C\lambda}\rangle = \sum_{\lambda}^{M} \langle n_{C\lambda}\rangle(1-\omega_\lambda^2) = \langle n_C\rangle(1-\omega^2).$$

Using (3.8*b*) we obtain the photon-counting distribution in the same way:

$$p(n) = \frac{1}{n!}\frac{d^n}{d(ix)^n}\prod_\lambda (1-ix\langle n_{T\lambda}\rangle)^{-1}\exp\left(\frac{ix\langle n_{C\lambda}\rangle}{1-ix\langle n_{T\lambda}\rangle}\right)\Bigg|_{ix=-1}$$

$$= \prod_\lambda (1+\langle n_{T\lambda}\rangle)^{-1}\exp\left(-\frac{\langle n_{C\lambda}\rangle}{1+\langle n_{T\lambda}\rangle}\right)\sum_{\Sigma_\lambda n_\lambda = n}\prod_\lambda \frac{1}{n_\lambda!}\left(1+\frac{1}{\langle n_{T\lambda}\rangle}\right)^{-n_\lambda}$$

$$\times L^0_{n_\lambda}\left\{-\frac{\langle n_{C\lambda}\rangle}{\langle n_{T\lambda}\rangle(\langle n_{T\lambda}\rangle+1)}\right\} \tag{3.12}$$

and putting $\langle n_{T\lambda}\rangle = 0$ for $\lambda = M+1, ..., 2M$ and $\langle n_{T\lambda}\rangle = \langle n_T\rangle/M$ for $\lambda = 1, ..., M$ and using (3.11) we arrive at (3.5) again if

$$\sum_\lambda^M \langle n_{C\lambda}\rangle \to \langle n_C\rangle\omega^2$$

and

$$\sum_{\lambda=M+1}^{2M} \langle n_{C\lambda}\rangle = \langle n_C\rangle(1-\omega^2).$$

If the occupation numbers in the chaotic field are different we obtain for the generating function of the superposition of chaotic and coherent fields with shifted frequencies

$$\langle \exp(ix\hat{n})\rangle_N = \prod_\lambda^M (1-ix\langle n_{T\lambda}\rangle)^{-1}\exp\left(\frac{ix\omega_\lambda^2\langle n_{C\lambda}\rangle}{1-ix\langle n_{T\lambda}\rangle}\right)\exp\{ix\langle n_{C\lambda}\rangle(1-\omega_\lambda^2)\}$$

$$= \prod_\lambda^M (1-ix\langle n_{T\lambda}\rangle)^{-1}\exp\left(\frac{ix\omega_\lambda^2\langle n_{C\lambda}\rangle}{1-ix\langle n_{T\lambda}\rangle}\right)\exp\{ix\langle n_C\rangle(1-\omega^2)\}. \tag{3.13}$$

For the moments $\langle W^k\rangle_N$ and the photon-counting distribution $p(n)$ we have

$$\langle W^k\rangle_N = k!\sum_{j=0}^{k}\sum_{\Sigma_\lambda^M m_\lambda = j}\prod_\lambda^M \frac{1}{m_\lambda!}\langle n_{T\lambda}\rangle^{m_\lambda} L^0_{m_\lambda}\left(-\frac{\langle n_{C\lambda}\rangle\omega_\lambda^2}{\langle n_{T\lambda}\rangle}\right)\frac{\{\langle n_C\rangle(1-\omega^2)\}^{k-j}}{(k-j)!} \tag{3.14}$$

and

$$p(n) = \prod_\lambda^M (1+\langle n_{T\lambda}\rangle)^{-1}\exp\left(-\frac{\langle n_{C\lambda}\rangle\omega_\lambda^2}{1+\langle n_{T\lambda}\rangle}\right)\exp\{-\langle n_C\rangle(1-\omega^2)\}$$

$$\times\sum_{j=0}^{n}\sum_{\Sigma_\lambda^M n_\lambda = j}\prod_\lambda^M \frac{1}{n_\lambda!}\left(1+\frac{1}{\langle n_{T\lambda}\rangle}\right)^{-n_\lambda} L^0_{n_\lambda}\left\{-\frac{\langle n_{C\lambda}\rangle}{\langle n_{T\lambda}\rangle(\langle n_{T\lambda}\rangle+1)}\right\}\frac{\{\langle n_C\rangle(1-\omega^2)\}^{n-j}}{(n-j)!} \tag{3.15}$$

respectively. The corresponding integrated intensity distribution $P_N(W)$ can in principle be determined from the moment sequence $\{\langle W^k\rangle_N\}$.

There is a physical interest in extending these results to the case of s ordering of field operators (e.g. for $s = -1$ we shall obtain the moment equation for the description of detection of the field by means of the quantum counters). This can be done by substituting (3.7) into the following equation (PH)

$$\langle W^k \rangle_s = \frac{k!}{(k+M-1)!}\left(\frac{1-s}{2}\right)^k \left\langle \mathrm{L}_k^{M-1}\left(\frac{2W}{s-1}\right)\right\rangle_N \tag{3.16}$$

and so we obtain

$$\begin{aligned}\langle W^k \rangle_s &= \frac{k!}{(k+M-1)!}\left(\frac{1-s}{2}\right)^k \\ &\times \sum_{j=0}^{k} \frac{1}{(j+M-1)!}\left(\frac{2\langle n_\mathrm{T}\rangle}{(1-s)M}\right)^j \mathrm{L}_j^{M-1}\left(-\frac{\langle n_\mathrm{C}\rangle \omega^2 M}{\langle n_\mathrm{T}\rangle}\right) \\ &\times \mathrm{L}_{k-j}^{j+M-1}\left(-\frac{2\langle n_\mathrm{C}\rangle(1-\omega^2)}{1-s}\right).\end{aligned} \tag{3.17}$$

The s-ordered moments $\langle W^k \rangle_s$ for arbitrary mean occupation numbers per mode can be obtained by substituting (3.9) into (3.16) or from the s-ordered generating function

$$\langle \exp(\mathrm{i}x\hat{n})\rangle_s = \prod_\lambda \left\{1-\mathrm{i}x\left(\langle n_{\mathrm{T}\lambda}\rangle + \frac{1-s}{2}\right)\right\}^{-1} \exp\left[\frac{\mathrm{i}x\langle n_{\mathrm{C}\lambda}\rangle}{1-\mathrm{i}x\{\langle n_{\mathrm{T}\lambda}\rangle+\frac{1}{2}(1-s)\}}\right] \tag{3.18}$$

which follows from (2.7). We obtain in this way

$$\langle W^k \rangle_s = k! \sum_{\Sigma_\lambda m_\lambda = k} \prod_\lambda \frac{1}{m_\lambda!}\left(\langle n_{\mathrm{T}\lambda}\rangle + \frac{1-s}{2}\right)^{m_\lambda} \mathrm{L}_{m_\lambda}^0\left(-\frac{\langle n_{\mathrm{C}\lambda}\rangle}{\langle n_{\mathrm{T}\lambda}\rangle+\frac{1}{2}(1-s)}\right). \tag{3.19}$$

A more general distribution than (2.12) (with arbitrary mean occupation numbers per mode) can be obtained from the sequence $\{\langle W^k\rangle_s\}$, i.e. from (3.19) using the Laguerre polynomials (Morse and Feshbach 1953, chap. 8, for some existence problems see Peřina and Mišta 1969, Peřina and Horák 1969 b, Peřina, to be published):

$$P(W,s) = W^{M-1}\exp(-W)\sum_{n=0}^{\infty} c_n \mathrm{L}_n^{M-1}(W) \tag{3.20}$$

where

$$c_n = \frac{n!}{(n+M-1)!}\sum_{j=0}^{n}\frac{(-1)^j}{j!(n-j)!(j+M-1)!}\langle W^j\rangle_s \tag{3.21}$$

under the assumption that

$$\int_0^\infty P^2(W,s)W^{1-M}\exp(W)\,\mathrm{d}W = \sum_{n=0}^{\infty}\frac{\{(n+M-1)!\}^3}{n!}|c_n|^2 < \infty. \tag{3.22}$$

4. Special cases

If we put $\omega = 1$ in (2.12) and use the relation

$$\mathrm{I}_m(x) \underset{x\to 0}{\simeq} \left(\frac{x}{2}\right)^m \frac{1}{\Gamma(m+1)} \tag{4.1}$$

we obtain (PH)

$$\begin{aligned}P(W,s) &= \left(\frac{\langle n_\mathrm{T}\rangle}{M}+\frac{1-s}{2}\right)^{-1}\left(\frac{W}{\langle n_\mathrm{C}\rangle}\right)^{(M-1)/2}\exp\left\{-\frac{W+\langle n_\mathrm{C}\rangle}{\langle n_\mathrm{T}\rangle/M+\frac{1}{2}(1-s)}\right\} \\ &\times \mathrm{I}_{M-1}\left\{2\frac{(\langle n_\mathrm{C}\rangle W)^{1/2}}{\langle n_\mathrm{T}\rangle/M+\frac{1}{2}(1-s)}\right\}\end{aligned} \tag{4.2}$$

where we have also used the identity (Morse and Feshbach 1953, chap. 6)

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{\{\Gamma(n+a+1)\}^2} \mathrm{L}_n^a(y) = (xy)^{-a/2} \exp(-x) \mathrm{I}_a\{2(xy)^{1/2}\}. \tag{4.3}$$

From (3.17) one can obtain (3.7) again by putting $s = 1$ and using the asymptotic formula for the Laguerre polynomial

$$\mathrm{L}_{k-j}^{j+M-1}.$$

Similarly, putting $\omega = 1$ ($\Omega = 0$) in (3.17) we obtain (PH)

$$\langle W^k \rangle_s = \frac{k!}{(k+M-1)!} \left(\frac{\langle n_\mathrm{T} \rangle}{M} + \frac{1-s}{2}\right)^k \mathrm{L}_k^{M-1}\left(-\frac{\langle n_\mathrm{C} \rangle}{\langle n_\mathrm{T} \rangle / M + \frac{1}{2}(1-s)}\right). \tag{4.4}$$

Further, equations (3.5) and (3.7) for $M = 1$ complete the third line of table 1 in the paper by Jakeman and Pike (1969). If we put $k = 2$ and $M = 1$ in (3.7) we arrive at equation (34) of Jakeman and Pike (1969):

$$\langle W^2 \rangle_N = 2\langle n_\mathrm{T} \rangle^2 + 2\langle n_\mathrm{T} \rangle \langle n_\mathrm{C} \rangle (1+\omega^2) + \langle n_\mathrm{C} \rangle^2. \tag{4.5}$$

Some other physically important results can be obtained from (2.12), (3.17) and (3.19) for $s = 0$ and $s = -1$, which correspond to the symmetric and anti-normal orderings of field operators respectively. The case $s = -1$ is physically meaningful in connection with the determination of the statistical properties of the fields considered by means of the quantum counters operating by stimulated emission rather than by absorption (Mandel 1966, Glauber, to be published). A number of new results can also be obtained by putting $\omega = 0$ ($\Omega \to \infty$) in the above equations. The corresponding equations for the coherent field can be obtained by putting $\langle n_{\mathrm{T}\lambda} \rangle = 0$ for all λ ($\langle n_\mathrm{T} \rangle = 0$) and for the chaotic field by putting $\langle n_{\mathrm{C}\lambda} \rangle = 0$ for all λ ($\langle n_\mathrm{C} \rangle = 0$).

5. Conclusion

In conclusion we should like to point out that the present analysis serves for the study of the statistical properties of the superposition of coherent and chaotic fields of an arbitrary spectrum (it can also serve for obtaining the formulae completing the first line of Jakeman and Pike's table). However, in this approach one has to consider some sampling points of the spectrum only. These questions and some possibilities of extending the present method to a continuous spectrum are under investigation now and will be dealt with in a forthcoming paper.

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